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Memory effects upon the relaxation dynamics of the Wigner distribution function

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Abstract. The solutions of the generalized Fokker-Planck equation that arises in the semiclassical description of a quantal harmonic coordinate immersed in an arbitrary heat bath are investigated in the non-Markovian regime. Our treatment consists of retaining a finite number of terms of the analytic expansion of the Laplace-transformed memory kernel around the origin which leads to approximate relaxation frequencies involving different degrees of memory. For a regime of highly inelastic collisions we extend previous findings concerning the relation between Markovian and non-Markovian relaxation frequencies and, for the usual elastic coupling, we consider fermion and phonon environments showing that the former involves a much more complex dynamics with a non-exponential time decay. Focusing upon the exponential part of the decay, we find that for both classes of reservoirs the first two non-Markovian corrections have well defined signs that increase the relaxation frequencies.

1. Introduction

The time evolution of damped quantum systems has been investigated with renewed interest in recent years. Among the best known approaches in the literature one may find the quantum Langevin equation [1, 2] arising from the Heisenberg picture, the master equation for the reduced density operator of the damped system [3-5], the formulation in terms of functional integral techniques (path integrals) [6, 7] and the quantum Fokker-Planck equation for the Glauber-Sudarshan quasiprobability distribution function [8, 9]. Regarding this latter approach, we have recently examined a different quantum Fokker-Planck equation fulfilled by the Wigner distribution function which avoids the well known drawbacks introduced by the singularities of Glauber's P-function [10]. Our investigation is concerned with the time evolution of a quantal harmonic mode immersed in a heat bath with which it interacts through a linear weak-coupling device.

In this configuration, we distinguish two very different kinds of memory effects. On the one hand, ergodicity causes the loss of memory of the initial conditions through a particular process: the system progressively forgets the highest energy moments of the initial distribution [5]. The second memory effect stems from the well known

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non-Markovian nature of the reduced equations of motion and, in this respect, we have derived a rule that relates the non-Markovian relaxation time to the Markovian one in a regime of highly inelastic collisions [3].

Along these lines, two main purposes orient the present paper: first, the study of the relaxation process of the Wigner distribution function for arbitrary environments and initial conditions and, secondly, the investigation of the above-mentioned memory effects. In order to carry out this programme, in section 2 we summarize the major results concerning the phase space mapping of the Markovian master equation [5] and its associated phonon dynamics. In section 3 we analyse a non-Markovian regime of highly inelastic collisions, for which we extend previous findings concerning the relation between Markovian and non-Markovian relaxation frequencies. The usual elastic non-Markovian regime is considered in section 4 where, assuming given regularity properties of the density of states in the thermodynamic limit, we calculate the coefficients of the analytic expansion of the Laplace-transformed memory kernel around the origin. This result is applied in section 5 to two specific heat reservoirs, an oscillator bath and a fermion fluid. Finally, in section 6 some concluding remarks are given.

2. The generalized Fokker–Planck equation and its solution

In [10] we have shown that the Wigner image of the master equation describing the relaxation of a quantal harmonic oscillator linearly coupled to a heat reservoir is

$$\frac{\partial\rho}{\partial t}(H,t)=\nu\left(\rho+(kT_{\mathrm{w}}+H)\frac{\partial\rho}{\partial H}+kT_{\mathrm{w}}H\frac{\partial^2\rho}{\partial H^2}\right) \tag{2.1}$$

where $\rho(H,t)$ is the semiclassical density, $H=p^2/2M+\frac{1}{2}M\Omega^2q^2$ is the semiclassical energy variable in phase space, the frequency parameter ν is the difference between the downwards and upwards transition rates in the master equation:

$$\begin{aligned}\nu&=W_+-W_-\\&=W_+(1-\beta)\end{aligned} \tag{2.2}$$

where

$$\beta=\exp(-\hbar\Omega/kT_{\mathrm{eff}}) \tag{2.3}$$

is the Boltzmann factor for an equilibrated oscillator with energy quantum $\hbar\Omega$ in a reservoir at an effective temperature [4] T_{eff}. The Wigner temperature T_{w} is the mean energy of the quantal oscillator at thermal equilibrium:

$$\begin{aligned}kT_{\mathrm{w}}&=\frac{\hbar\Omega}{2}\,\frac{1+\beta}{1-\beta}\\&=\frac{\hbar\Omega}{2}\coth\frac{\hbar\Omega}{2kT_{\mathrm{eff}}}.\end{aligned} \tag{2.4}$$

The general solution of equation (2.1) reads [10]

$$\rho(h,\tau)=\rho_0(h)\left[1+\sum_{m\geq 1}a_m\exp\left(-\frac{m\tau}{\vartheta+\frac{1}{2}}\right)L_m\left(\frac{h}{\vartheta}\right)\right] \tag{2.5}$$

where the expansion coefficients a_m are obtained from initial conditions, $\rho_0(h)$ is the equilibrium density,

$$\rho_0(h)=\frac{1}{\vartheta}\exp\left(-\frac{h}{\vartheta}\right) \tag{2.6}$$

the function $L_m(x)$ is the mth Laguerre polynomial and we have chosen the adimensional variables $\vartheta = kT_w/\hbar\Omega$, $h = H/\hbar\Omega$ and $\tau = W_+t$.

Let us now consider the nth moment of the energy-adimensionalized Fokker–Planck equation:

$$M_n(t) = \int_0^\infty \mathrm{d}h\, h^n \rho(h, t). \tag{2.7}$$

A simple calculation leads to the coupled moment equations:

$$\dot{M}_n(t) = \nu(-nM_n(t) + n^2\vartheta M_{n-1}(t)). \tag{2.8}$$

The analysis of the spectrum of this vector equation immediately yields the eigenvalues $\lambda_k = -k\nu$, $k = 0, 1, \ldots$, and the eigenvectors $\mu^{(k)}$ with components

$$\mu_n^{(k)} = \begin{cases} 0 & n < k \\ (-)^k \binom{n}{k} \vartheta^n n! & n \geq k. \end{cases} \tag{2.9}$$

In other words, the nth moment evolves according to the n-lowest eigenfrequencies of the Fokker–Planck equation (2.1). This point is in agreement with a previous result obtained in the quantum case [5] and reflects the fact that Wignerization is a linear procedure. One can further verify that the solutions of equation (2.8) possess the asymptotic limit

$$M_n(\infty) = n!\,\vartheta^n \tag{2.10}$$

which corresponds to the classical canonical distribution (2.6), with ϑ being just the asymptotic value of the first moment:

$$\vartheta = \tfrac{1}{2} + \frac{\beta}{1-\beta}. \tag{2.11}$$

3. The non-Markovian semiclassical equation

The non-Markovian version of the generalized Fokker–Planck equation (2.1), in terms of the adimensional energy variable h and energy parameter ϑ, is [10]

$$\frac{\partial\rho}{\partial t} = \int_0^t \mathrm{d}\tau\, \nu(\tau) \left(\rho + (\vartheta + h)\frac{\partial\rho}{\partial h} + \vartheta h \frac{\partial^2\rho}{\partial h^2} \right)_{t-\tau}. \tag{3.1}$$

In this equation, both the frequency density ν and the Wigner temperature $kT_w = \vartheta\hbar\Omega$ acquire some time dependence that should be traced to the variations of the microscopic transition rates $W_\pm(\tau)$ [10]. As usual, the convolution in (3.1) decouples by means of the Laplace transformation,

$$\rho(h, \lambda) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\lambda t} \rho(h, t) \tag{3.2}$$

and we find the transformed evolution equation,

$$\vartheta(\lambda) h \frac{\partial^2\rho(h,\lambda)}{\partial h^2} + (h + \vartheta(\lambda))\frac{\partial\rho(h,\lambda)}{\partial h} + \left(1 - \frac{\lambda}{\nu(\lambda)}\right)\rho(h,\lambda) + \frac{\rho(h, t=0)}{\nu(\lambda)} = 0. \tag{3.3}$$

In equation (3.3), the functions $\nu(\lambda)$ and $\vartheta(\lambda)$ are

$$\nu(\lambda)=W_+(\lambda)-W_-(\lambda)$$
$$=W_+(\lambda)(1-\beta(\lambda)) \tag{3.4a}$$

$$\vartheta(\lambda)=\frac{1}{2}\frac{1+\beta(\lambda)}{1-\beta(\lambda)} \tag{3.4b}$$

with $W_\pm(\lambda)$ the Laplace transforms of the time-dependent rates $W_\pm(\tau)$.

It has been shown in [10] that a general solution for (3.3) exists and is of the form

$$\rho(h,\lambda)=\frac{e^{-h/\vartheta}}{\vartheta}\left(\frac{1}{\lambda}+\sum_{m=1}^{\infty}\frac{1}{\lambda+m\nu(\lambda)}a_m(\lambda)L_m\left(\frac{h}{\vartheta}\right)\right) \tag{3.5}$$

where the expansion coefficients a_m (equation (2.5)) acquire a λ dependence arising from the function $\vartheta(\lambda)$. The non-Markovian eigenfrequencies are then the roots of the nonlinear equation [4, 5]

$$\lambda_m=-m\nu(\lambda_m). \tag{3.6}$$

The determination of these roots, which might be expected to be complex numbers, in general, demands a model for the coupling that drives the oscillator towards thermal equilibrium. A definite model plus some assumptions regarding the characteristics of the interaction—i.e. weak coupling—yield detailed expressions for the transition rates $W_\pm(\lambda)$. In previous works [3-5, 10], we have considered an equilibrated fermionic heat bath and analysed the nature of the non-Markovian eigenfrequencies in the highly inelastic limit. The Markovian approximation appears to be valid in the weakly coupled, overdamped situation. In particular, in [3] we have derived a rule that permits one to relate the lifetime of an excitation in the non-Markovian regime, namely the smallest root λ_1, to the Markovian counterpart $\lambda_1^{(M)}$. We are now interested in looking for a generalization of the above rule to every eigenfrequency (3.6) whose Markovian counterpart is

$$\lambda_m^{(M)}=-m\nu(0). \tag{3.7}$$

Assuming that both $\lambda_0=0$ and λ_m belong to the regularity domain of $\nu(\lambda)$ in the complex plane [11, 12], we may write

$$\nu(\lambda_m)=\nu(0)+\nu'(0)\lambda_m+\tfrac{1}{2}\nu''(0)\lambda_m^2+\dots \tag{3.8}$$

and then a sufficient condition for the Markovian rule (3.7) to hold is

$$|\nu^{(\iota)}(0)\lambda_m^{\iota}|\ll\nu(0) \tag{3.9}$$

for all positive integers ι. The weakest non-Markovian regime arises if second- and higher-order terms in expansion (3.8) are neglected, yielding in this case the approximation

$$\lambda_m\approx-\frac{m\nu(0)}{1+m\nu'(0)}$$
$$=\frac{\lambda_m^{(M)}}{1+m\nu'(0)}. \tag{3.10}$$

On the other hand, it has been shown [10] that in the weak-coupling limit that permits a golden-rule description of the scattering rates, the complex transition probabilities $W_\pm(\lambda)$ are of the form

$$W_\pm(\lambda)=\sum_P \frac{\gamma+\lambda}{(\gamma+\lambda)^2+\omega_P^2}\, W_\pm^P. \tag{3.11}$$

In this expression, γ is an inelasticity spread related to the inverse correlation time, $\gamma\approx\tau_{\mathrm{corr}}^{-1}$, ω_P is the transition energy for the event P (which is in turn labelled by a set of r quantum numbers, $r\geqslant 1$) and $W_\pm^P$ are amplitudes that depend on the squared interaction strength and on the equilibrium distribution of the heat reservoir. We then have,

$$\nu(\lambda)=\sum_P \frac{\gamma+\lambda}{(\gamma+\lambda)^2+\omega_P^2}\, W_P \tag{3.12}$$

with $W_P=W_+^P-W_-^P>0$.

A reasonable estimate for the appearance of the function $\nu(\lambda)$ can be given as follows. On the one hand, assuming a large heat bath a thermodynamic limit holds for any function α_P:

$$\sum_P \alpha_P \rightarrowtail \int \alpha(\rho)g(\rho)\,\mathrm{d}^r\rho \tag{3.13}$$

where $\mathrm{d}^r\rho$ is the r-dimensional differential label denoting the scattering process ρ and $g\ (\rho)$ is the density of states in the Hilbert space of the reservoir channels. We then express equation (3.12) as

$$\nu(x)=x\int \frac{\mathrm{d}^r\rho\ W(\rho)g(\rho)}{x^2+\omega^2(\rho)} \tag{3.14}$$

with $x=\gamma+\lambda$. On the other hand, we may generally find a variable change from the r ρ-labels to a set having $\omega\ (\rho)$ as a member. One can then perform the uninteresting integrals and obtain,

$$\nu(x)=x\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega\, F(\omega)}{x^2+\omega^2}. \tag{3.15}$$

Usually, conservation laws lead to a peaked function $F(\omega)$ around the origin (cf equations (4.6) and (4.7) of [4]); therefore, the Breit–Wigner filter in equation (3.15) may be approximated, restricting the integration domain to an interval $[-\mu x, \mu x]$, μ being a number close to unity. Accordingly,

$$\nu(x)\approx\frac{1}{x}\int_{-x}^{x} \mathrm{d}\omega\, F(\omega) \tag{3.16}$$

and, as a consequence,

$$\nu'(x)\approx -\frac{\nu}{x}+\frac{2F_{\mathrm{even}}(x)}{x} \tag{3.17a}$$

$$\nu''(x)\approx\frac{2\nu(x)}{x^2}-\frac{4F_{\mathrm{even}}(x)}{x^2}+\frac{2F'_{\mathrm{even}}(x)}{x} \tag{3.17b}$$

with

$$F_{\mathrm{even}}(x)=\tfrac{1}{2}(F(x)+F(-x)). \tag{3.18}$$

Let us truncate expansion (3.8) up to second order and replace into the definition (3.6) of the mth roots. One easily finds a second-degree equation:

$$(\lambda_m/\gamma)^2(\lambda_m^{(M)}+2m\Phi-m\gamma\Phi')-(\lambda_m/\gamma)(\gamma+\lambda_m^{(M)}+2m\Phi)+\lambda_m^{(M)}=0 \tag{3.19}$$

with $\Phi=F_{\text{even}}(\gamma)$. Now, one knows that the mth lifetime is the smallest of the two roots [4, 5, 10]; it can be then seen that, for those values of m for which $|\lambda_m|$, $|\lambda_m^{(M)}|$ are much smaller than γ, the smallest solution of equation (3.19) can be approximated by

$$\frac{\lambda_m}{\gamma}\approx\frac{\lambda_m^{(M)}}{\gamma+\lambda_m^{(M)}+2m\Phi} \tag{3.20}$$

which also arises from equations (3.10) and (3.17a).

This expression generalizes that obtained in [3] for the so-called lifetime of the quantal harmonic excitation, namely the smallest λ_1 root. Notice, however, that it is not valid for sufficiently large m, which is not important, since roots λ_m close to γ represent microscopic lifetimes and are irrelevant to any study of macroscopic relaxation.

4. The non-Markovian regime under an elastic coupling

In the preceding section we have focused upon the highly inelastic limit occurring when the non-Markovian relaxation frequencies are much smaller than the inelasticity spread γ. One may call such a regime a 'weakly non-Markovian' one, since the relaxation spectrum is close to that given by the Markovian prescription. We are now attempting to study an analogous weakly non-Markovian regime in the case of a vanishing inelasticity $\gamma=0$. Firstly, we may write equation (3.15) for $\gamma=0$ as follows:

$$\begin{aligned}\nu(\lambda)&=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega\,F(\omega)}{\lambda+\mathrm{i}\omega}+\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega\,F(\omega)}{\lambda-\mathrm{i}\omega}\\&=-\mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega\,F_{\text{even}}(\omega)}{\omega-\mathrm{i}\lambda}\end{aligned} \tag{4.1}$$

with $F_{\text{even}}(x)$ given in equation (3.18). In order to analyse the above expression it is convenient to change variables to

$$z=\mathrm{i}\lambda \tag{4.2}$$

leading to

$$\varphi(z)\equiv\frac{\nu(-\mathrm{i}z)}{2\pi}=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega\,F_{\text{even}}(\omega)}{\omega-z}. \tag{4.3}$$

Thus the RHS of equation (4.3) is just of the form of a Cauchy integral [13, 14] which generally gives rise to a regular function $\varphi(z)$ in the upper half-plane that approaches zero for $z\rightarrowtail\infty$.

This behaviour can be understood since $\nu(\lambda)$ is a Laplace transform of the type (3.2), i.e. defined for Re $\lambda>0$. However, the non-Markovian frequencies arising as the roots of equation (3.6) must be located in the left half-plane and thus one has to compute the analytic continuation of equation (4.3) in the lower half-plane. This problem may be easily solved if the function $F_{\text{even}}(\omega)$, which is initially defined on the real axis, can be analytically extended beyond that axis, simply replacing the real variable ω by a complex z [14]. Then assuming $F_{\text{even}}(z)$ to be a regular function in a circle of radius R centred at the origin $z=0$, the analytic continuation of equation (4.3) reads [11-15]

$$\varphi(z)=-\varphi(-z)+F_{\text{even}}(z) \qquad \operatorname{Im} z<0 \tag{4.4a}$$

$$\varphi(z)=\frac{1}{2\pi\mathrm{i}}\mathscr{P}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{F_{\text{even}}(\omega)}{\omega-z}+\tfrac{1}{2}F_{\text{even}}(z) \qquad z \text{ real} \tag{4.4b}$$

where the function φ on the RHS of equation (4.4a) is given by expression (4.3) and the symbol $\mathscr{P}$ in equation (4.4b) stands for the Cauchy principal part.

We must remark that the analytic continuation formulae (4.4) are valid in the regularity domain of $F_{\text{even}}(z)$, i.e. in a half-circle of radius R around the origin in the lower half-plane. In the border of such a half-circle there should be at least one singularity of $F_{\text{even}}(z)$ and in this respect one may distinguish two main cases:

(i) singularities located on the real axis [11, 12]. They generally arise from cut-off frequencies of the function $F_{\text{even}}(\omega)$ like those appearing in a Debye-like density in a phonon reservoir. In addition, it can be shown [13] that these points are, in general, branching points of the Cauchy integrals in equations (4.3) and (4.4).

(ii) If $F_{\text{even}}(z)$ is regular along the whole real axis, it must necessarily possess singularities outside this axis, otherwise the function $\varphi(z)$ defined by equations (4.3) and (4.4) would identically vanish [14].

In any case one finds [11, 12] that the singularities of F_{even} are correlation, i.e. microscopic, frequencies and, therefore, the inverse of the radius R usually gives the characteristic correlation lifetime.

The coefficients of the series expansion (3.8) can be explicitly written in the current weak-coupling regime. In fact, assuming a weight function $F(x)$ differentiable along the whole real axis, a straightforward calculation yields the nth derivative of $\varphi(z)$ in $z=0$ [15], and thus we have

$$\nu^{(n)}(0)=\begin{cases}(-)^{n/2}\pi F_{\text{even}}^{(n)}(0)=(-)^{n/2}\pi F^{(n)}(0) & n \text{ even} \quad (4.5a)\\[2ex] (-)^{(n-1)/2}n!\displaystyle\int_{-\infty}^{\infty}\mathrm{d}\omega\,F_n(\omega) & n \text{ odd} \quad (4.5b)\end{cases}$$

where we have defined the set of functions

$$F_n(\omega)=\frac{1}{\omega^{n+1}}\left(F_{\text{even}}(\omega)-\sum_{k=0}^{(n-1)/2}\frac{\omega^{2k}}{(2k)!}F_{\text{even}}^{(2k)}(0)\right) \tag{4.6}$$

for positive odd n.

One can verify that the above functions possess the same spectrum of singularities as $F_{\text{even}}(z)$. Then, assuming that such a spectrum consists only of the poles z_k and that $F_{\text{even}}(z)/z$ approaches zero for $z\rightarrowtail\infty$, the integral (4.5b) yields

$$\nu^{(n)}(0)=(-)^{(n-1)/2}n!2\pi\mathrm{i}\sum_{\operatorname{Im} z_k>0}\operatorname{Res}[F_n, z_k] \qquad n \text{ odd}. \tag{4.5c}$$

In the next section we will apply the above formalism to the case of a well known system, the phonon reservoir, and to a lesser known coupling, the fermion heat bath.

5. Elastic coupling to specific reservoirs

5.1. Phonon reservoir

Let us assume that the harmonic oscillator is linearly coupled to a phonon reservoir through the Hamiltonian

$$H_{\text{int}}=\sum_{\alpha}\Gamma_{\alpha+}a^{\dagger}a_{\alpha}+\Gamma_{\alpha-}a^{\dagger}a_{\alpha}^{\dagger}+\text{HC} \tag{5.1}$$

where $a^{\dagger}$, $a_{\alpha}^{\dagger}$ represent phonon creation operators of the central oscillator and mode α respectively and HC denotes the Hermitian conjugate terms. Such a Hamiltonian represents the most general linear coupling and some usual particular choices of the matrix elements $\Gamma_{\alpha\pm}$ are:

(i) $\Gamma_{\alpha+}=\Gamma_{\alpha-}$, representing an interaction of the coordinate-coordinate type [1, 2, 16, 17];

(ii) $\Gamma_{\alpha-}=0$, representing the so-called rotating-wave approximation [1, 2].

A straightforward calculation [8] leads to the Laplace-transformed transition rates:

$$W_{\pm}(\lambda)=\frac{2}{\hbar^2}\sum_{\alpha}|\Gamma_{\alpha\mp}|^2 n_{\alpha}\frac{\lambda}{\lambda^2+(\Omega\pm\omega_{\alpha})^2}+|\Gamma_{\alpha\pm}|^2(n_{\alpha}+1)\frac{\lambda}{\lambda^2+(\Omega\mp\omega_{\alpha})^2} \tag{5.2}$$

where n_{α} symbolizes the mean number of phonons of frequency ω_{α}. We notice that, in any case, $\nu(\lambda)=W_{+}(\lambda)-W_{-}(\lambda)$ is independent of n_{α}, i.e. of the temperature; for simplicity, we will restrict ourselves to a coordinate-coordinate interaction, taking $\Gamma_{\alpha+}=\Gamma_{\alpha-}=\Gamma_{\alpha}$. Thus, turning to the continuum limit (cf equation (3.13)),

$$\sum_{\alpha}\frac{\Gamma_{\alpha}^2}{\hbar^2}\rightarrowtail\int_0^{\infty}f(\omega)\,\mathrm{d}\omega \tag{5.3}$$

it is easy to find the following expression for the function $\varphi(z)=\nu(-\mathrm{i}z)/2\pi$ in the upper half-plane:

$$\varphi(z)=\frac{1}{2\pi\mathrm{i}}\left(\int_{-\infty}^{\Omega}f(-\omega+\Omega)-\int_{\Omega}^{\infty}f(\omega-\Omega)-\int_{-\infty}^{-\Omega}f(-\omega-\Omega)+\int_{-\Omega}^{\infty}f(\omega+\Omega)\right)$$
$$\times\frac{\mathrm{d}\omega}{\omega-z}. \tag{5.4}$$

At this point we may observe that as ω approaches zero, the weight function $f(\omega)$ must vanish at least as ω itself, in order to keep finite the RHS of equation (5.2). Thus the condition $f(0)=0$ makes room for a possible analytic extension of $f(\omega)$ for $\omega<0$ by means of an odd parity rule, which would carry equation (5.4) into equation (4.3) with

$$F_{\text{even}}(\omega)=f(\Omega+\omega)+f(\Omega-\omega). \tag{5.5}$$

In order to get more explicit results we will focus upon the weight function [1, 8, 16, 17]

$$f(\omega)=\frac{\Lambda}{\pi}\frac{\eta^2\omega}{\eta^2+\omega^2} \tag{5.6}$$

with η representing the phonon bandwidth of the reservoir excitations that may couple to the oscillator and with Λ being an adimensional average strength of such a coupling.

An analysis of the function (5.5) shows that it possesses four single poles with real part $\pm\Omega$ and imaginary part $\pm\eta$ which yield a convergence radius,

$$R=\sqrt{\Omega^2+\eta^2} \tag{5.7}$$

for the series expansion (3.8). The first three coefficients of this expansion can be computed by means of equations (4.5*a*) and (4.5*c*), giving

$$\nu(0)=2\Lambda\Omega\eta^2/R^2 \tag{5.8a}$$

$$\nu'(0)=-4\Lambda\Omega\eta^3/R^4 \tag{5.8b}$$

$$\nu''(0)=12\Lambda\Omega\eta^2(\eta^2-\Omega^2/3)/R^6. \tag{5.8c}$$

Thus, assuming that the oscillator frequency Ω is smaller than the phonon bandwidth η [1, 17] one may realize that the general term of the expansion (3.8) is approximately given by

$$\nu^{(n)}(0)\lambda_m^n\sim\Lambda\Omega(\lambda_m/R)^n. \tag{5.9}$$

According to equation (5.9), we see that the Markovian limit (3.9) holds if $|\lambda_m|\ll R$, leading us to identify the convergence radius R as the characteristic correlation frequency as expected.

5.2. *Fermion reservoir*

Let us now consider the coupling to a fermion heat bath through the Hamiltonian [3-5]

$$H_{\text{int}}=\sum_{\alpha\mu}\Lambda_{\alpha\mu}a^{+}b_\mu^{\dagger}b_\alpha+\text{HC} \tag{5.10}$$

where $b_\mu^{\dagger}$ (b_α) represents the fermion creation (destruction) operator of the single-particle state $|\mu\rangle$ ($|\alpha\rangle$) and $\Lambda_{\alpha\mu}$ symbolizes the associated matrix element.

The Laplace-transformed transition rates corresponding to the above Hamiltonian have been calculated in [3, 4], and read

$$W_\pm(\lambda)=\frac{2g^2}{\hbar^2}\sum_{\alpha\mu}|\Lambda_{\alpha\mu}|^2\frac{\lambda}{\lambda^2+(\Omega-\omega_{\alpha\mu})^2}\begin{Bmatrix}\rho_\mu(1-\rho_\alpha)\\ \rho_\alpha(1-\rho_\mu)\end{Bmatrix} \tag{5.11}$$

where $\rho_A=[1+\exp(\varepsilon_A-\varepsilon_F)/kT]^{-1}$ denotes the Fermi occupation number for a state $|A\rangle$ of energy ε_A, $\omega_{\alpha\mu}$ is the difference $(\varepsilon_\alpha-\varepsilon_\mu)/\hbar$ and g is a degeneracy factor related to internal fermion coordinates. Assuming a translationally invariant interaction, turning to the continuum limit according to the well known prescription

$$\sum_\mu\to(L/2\pi)^3\int \mathrm{d}^3k_\mu=\frac{L^3}{4\pi^2}\int_0^\infty k_r\,\mathrm{d}k_r\int_{+\infty}^{\infty}\mathrm{d}k_z \tag{5.12}$$

(L^3 denotes the volume of the fermion reservoir) and following the steps discussed in [4], we can show that $\nu(\lambda)$ takes the form of equation (3.15) with

$$F(\omega)=4\pi(g\Lambda)^2(L/h)^3(2m^3\varepsilon_F/3\hbar^2)^{1/2}kT/\hbar\Omega$$
$$\times\ln\left(\frac{1+\exp\{(\varepsilon_f/kT)(1/3\Omega^2)[3\Omega^2-(\omega+\Omega_-)^2]\}}{1+\exp\{(\varepsilon_F/kT)(1/3\Omega^3)[3\Omega^2-(\omega+\Omega_+)^2]\}}\right) \tag{5.13}$$

where m is the fermion mass, $\Lambda^2\equiv|\Lambda_{\alpha\mu}|^2$ and $\Omega=c_sq$, c_s being the sound velocity and $\hbar q$ the phonon momentum. We have defined the shifted frequencies ($\hbar\Omega<\varepsilon_F$),

$$\Omega_\pm=\Omega(1\pm3\hbar\Omega/4\varepsilon_F) \tag{5.14}$$

and have assumed low temperatures, so that both the fermion chemical potential $\varepsilon_F(T)$ and the sound velocity are well approximated by their zero-temperature values. Under this condition we have $c_s = (2\varepsilon_F/3m)^{1/2}$ [18].

According to equation (4.4*a*), the singularities of $\varphi(z)$ are those of $F_{\text{even}}(z)$ in the lower half-plane; from equations (3.18) and (5.13) we find that they group into eight sets of branch points given by

$$\omega = \Omega\{(-)^\eta\sqrt{3}\sqrt{1-(-)^\eta \mathrm{i}(2n+1)\pi kT/\varepsilon_F} + (-)^\zeta\Omega_\pm/\Omega\}$$
$$n = 0, 1, 2, \ldots (\eta, \zeta = 0 \text{ or } 1). \tag{5.15}$$

The branch points of the right (left) half-plane lie along four curves parallel to $\sqrt{3(1-\mathrm{i}u)}$ $(-\sqrt{3(1+\mathrm{i}u)})$, $u>0$, that is, they are symmetrically located with respect to the imaginary axis. Issuing from each branch point, the simplest choice for a cut ending at $\mathrm{Im}\, z = -\infty$ is the curve to which the given point belongs (figure 1).

The complexity of the above singularity spectrum is remarkable compared to the phonon reservoir, which exhibits only two temperature-independent poles. In addition, from equation (3.5) we realize that $\rho(h, \lambda = -\mathrm{i}z)$ displays the same singularity spectrum (apart from the poles arising from equation (3.6)); accordingly, non-exponential time-decaying terms are expected to contribute to $\rho(h, t)$ [12, 15, 17].

In the current low temperature limit:

(i) the eight curves mentioned above are indeed well defined ones, since the distance between branch points is of order kT/ε_F;

(ii) the convergence radius R is well approximated by $\sqrt{3}\,\Omega - \Omega_+$, i.e. it is of order Ω;

(iii) for z within the convergence radius ($|z| < \Omega$) we have from equation (5.13) a z-independent $F_{\text{even}}(z)$:

$$F_{\text{even}}(z) \approx 4\pi(g\Lambda)^2(L/h)^3(2m^3\varepsilon_F/3\hbar^2)^{1/2}. \tag{5.16}$$

This immediately leads to an odd-parity rule for the analytic continuation (4.4*a*) of $\varphi(z)$, and, accordingly, to

$$\nu(-\lambda) = 2\nu(0) - \nu(\lambda) \qquad (|\lambda| < R). \tag{5.17}$$

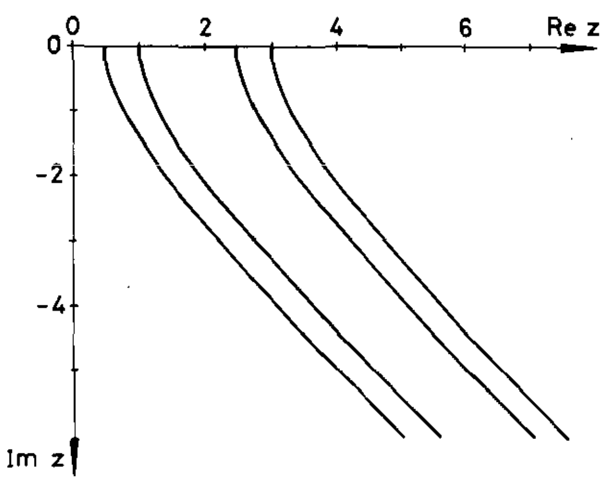

Figure 1. Schematic plot of the four curves parallel to $\sqrt{3(1-\mathrm{i}u)}$ $(u>0)$. The axes units are Ω and we have set $\hbar\Omega/\varepsilon_F = 0.339$ as in [4].

In [5] we have analysed the function $\nu(\lambda)$ for $\lambda>0$ and have found that it is practically temperature independent (for $0\leq kT/\varepsilon_F\leq\frac{1}{6}$). Thus, it is of interest to consider the case $T=0$ for which a calculation similar to that leading to equation (3.10) of [4] yields, in the present case,

$$\nu(\lambda)=\left(\frac{g\Lambda}{h}\right)^2\left(\frac{L}{\sqrt{3}\,\Omega}\right)^3\left(\frac{2m\varepsilon_F}{\hbar^2}\right)^{3/2}\left\{(\lambda^2+3\Omega^2-\Omega_-^2)\right.$$
$$\times\left[\tan^{-1}\left(\frac{\sqrt{3}\,\Omega-\Omega_-}{\lambda}\right)+\tan^{-1}\left(\frac{\sqrt{3}\,\Omega+\Omega_-}{\lambda}\right)\right]$$
$$-(\lambda^2+3\Omega^2-\Omega_+^2)\left[\tan^{-1}\left(\frac{\sqrt{3}\,\Omega-\Omega_+}{\lambda}\right)+\tan^{-1}\left(\frac{\sqrt{3}\,\Omega+\Omega_+}{\lambda}\right)\right]$$
$$\left.+\lambda\Omega_-\ln\left[\frac{\lambda^2+(\sqrt{3}\,\Omega+\Omega_-)^2}{\lambda^2+(\sqrt{3}\,\Omega-\Omega_-)^2}\right]+\lambda\Omega_+\ln\left[\frac{\lambda^2+\sqrt{3}\,\Omega-\Omega_+)^2}{\lambda^2+(\sqrt{3}\,\Omega+\Omega_+)^2}\right]\right\}. \tag{5.18}$$

The only singularities of the above function in the z-plane are eight branch points located on the real axis and given by equation (5.15) with $T=0$; in fact, as kT/ε_F goes to zero, the branch points of each curve merge to a single cut as illustrated in figure 1.

The first two coefficients of expansion (3.8) computed from equation (5.18) give

$$\nu(0)=4\pi^2(g\Lambda)^2(L/h)^3(2m^3\varepsilon_F/3\hbar^2)^{1/2} \tag{5.19a}$$

and

$$\nu'(0)=2\left(\frac{g\Lambda}{h}\right)^2\left(\frac{L}{\Omega}\right)^3\left(\frac{2m\varepsilon_F}{3\hbar^2}\right)^{3/2}\left\{\Omega_-\ln\left[\frac{\sqrt{3}\,\Omega+\Omega_-}{\sqrt{3}\,\Omega-\Omega_-}\right]+\Omega_+\ln\left[\frac{\sqrt{3}\,\Omega-\Omega_+}{\sqrt{3}\,\Omega+\Omega_+}\right]\right\} \tag{5.19b}$$

which is a negative quantity since $\Omega_+>\Omega_-$. The second derivative vanishes, in agreement with equation (4.5a) for $n=2$ under approximation (5.16); notice that the exact expression (5.13) yields the positive result

$$\nu''(0)=\left(\frac{2\pi g\Lambda}{\hbar}\right)^2\left(\frac{L}{h\Omega}\right)^3\frac{m^{3/2}}{kT}\left(\frac{2\varepsilon_F}{3}\right)^{5/2}e^{-\varepsilon_F/kT}$$
$$\times\left\{\left(\frac{\Omega_+}{\Omega}\right)^2\exp\left[\left(\frac{\Omega_+}{\Omega}\right)^2\frac{\varepsilon_F}{3kT}\right]-\left(\frac{\Omega_-}{\Omega}\right)^2\exp\left[\left(\frac{\Omega_-}{\Omega}\right)^2\frac{\varepsilon_F}{3kT}\right]\right\} \tag{5.19c}$$

which vanishes in the limit $kt/\varepsilon_F\to0$.

A general situation of higher temperature requires a numerical task which we leave for a future work. However, it is simple and instructive to analyse the high-temperature limit $\hbar\Omega<\varepsilon_F(T=0)\ll kT$ at which the Fermi occupation numbers in equation (5.11) become Maxwellian distributions and the sound velocity is given by $c_s=(\gamma kT/m)^{1/2}$, $(\gamma=C_P/C_V)$. In such a case, it is easy to find

$$F_{\text{even}}(\omega)=\left(\frac{g\Lambda}{\hbar}\right)^2\frac{N}{\pi\Omega}(2\pi\gamma)^{1/2}e^{-\gamma/2}e^{-(\omega/\Omega)^2\gamma/2}\left[e^{-\gamma\omega/\Omega}\sinh\left(\frac{\hbar\Omega}{2kT}\frac{\Omega+\omega}{\Omega}\right)\right.$$
$$\left.+e^{\gamma\omega/\Omega}\sinh\left(\frac{\hbar\Omega}{2kT}\frac{\Omega-\omega}{\Omega}\right)\right] \tag{5.20}$$

where N denotes the number of particles. The above function is regular in the whole lower half-plane except at $\mathrm{Im}\,\omega=-\infty$; this can be understood from the behaviour of equation (5.15) for higher temperatures. Thus, in this temperature limit the convergence radius R is indeed infinite. This does not imply a vanishing correlation lifetime; in fact, one finds

$$\nu(\tau)=W_+(\tau)-W_-(\tau)\sim \mathrm{e}^{+(\Omega\tau)^2/2\gamma}\sin(\Omega\tau)\sin\left(\frac{\hbar\Omega}{2\gamma kT}\Omega\tau\right) \tag{5.21}$$

i.e. a lifetime of order Ω^{-1}. Furthermore, for ω along the real axis we can replace the hyperbolic sines in equation (5.20) by their arguments and then, for real ω,

$$F_{\text{even}}(\omega)\simeq\frac{(g\Lambda)^2}{\pi}\frac{N}{\hbar kT}(2\pi\gamma)^{1/2}\,\mathrm{e}^{-\gamma/2}\,\mathrm{e}^{-(\omega/\Omega)^2\gamma/2}\left[\cosh\left(\frac{\gamma\omega}{\Omega}\right)-\frac{\omega}{\Omega}\sinh\left(\frac{\gamma\omega}{\Omega}\right)\right]. \tag{5.22}$$

From the above expression and equation (4.5) it is easy to extract the coefficients of expansion (3.8):

$$\nu(0)=(g\Lambda)^2\frac{N}{\hbar kT}(2\pi\gamma)^{1/2}\,\mathrm{e}^{-\gamma/2} \tag{5.23a}$$

$$\nu'(0)=\frac{\nu(0)}{\pi\Omega}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2}\,[\mathrm{e}^{-\gamma x^2/2}(\cosh\gamma x-x\sinh\gamma x)-1]<0 \tag{5.23b}$$

$$\nu''(0)=\frac{\nu(0)}{\Omega^2}\gamma(3-\gamma)>0 \tag{5.23c}$$

where the inequalities arise from the ideal gas relationship $1<\gamma<\frac{5}{3}$.

6. Concluding remarks

In this work we have investigated the non-Markovian time evolution of the semiclassical distribution for a harmonic oscillator linearly coupled to a heat bath compared to the Markovian case. To this aim we have examined first a highly inelastic limit, finding a generalization of the rule that relates Markovian and non-Markovian relaxation frequencies.

We have first illustrated our formalism for the usual elastic coupling in a well known situation, namely that of a harmonic oscillator heat bath. Next we investigated a different and less explored environment, that is, a fermion reservoir. We have found that this system exhibits a much more complex dynamics, in particular the singularity spectrum of the Laplace-transformed memory kernel is strongly dependent upon temperature and the nature of the singularities leads to terms of non-exponential time decay in the evolution of the damped oscillator. We have focused our investigation upon the exponential part of the decay, exploring the poles of the Laplace-transformed Wigner distribution function and explicit results have been derived for both limits of temperature with emphasis in the weak non-Markovian regime represented by the first- and second-order corrections to the Markovian relaxation frequencies. Regarding this point, it is very remarkable that for both classes of reservoirs the non-Markovian corrections have well defined signs that increase the relaxation frequencies, as is schematically illustrated in figure 2. This feature may then indicate a possible generaliz-

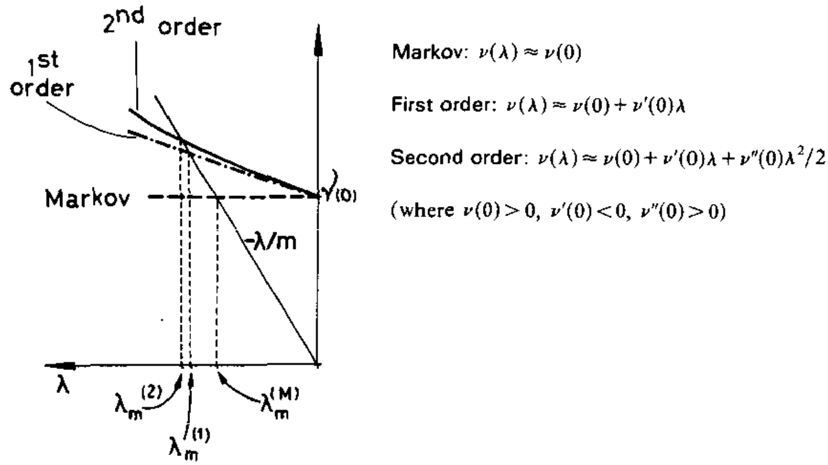

Figure 2. Graphic solution of equation (3.6) for Markov, first- and second-order approximations.

ation of the well known 'universality' of the Markovian weak-coupling regime [16, 17]. Finally, the effect of non-exponential contributions on the relaxation process remains to be discussed. We have seen that such terms arise from the branch points occurring on the frequency plane, thus their time evolution should be governed by the lowest branch point, i.e. the convergence radius R. On the other hand, in the weak non-Markovian regime we have focused upon the lowest pole λ_1 which governs the exponential behaviour must fulfil $|\lambda_1| \ll R$ (the Markovian limit corresponds to $R \to \infty$). This good separation of timescales should produce a rapid convergence of non-exponential terms to negligible values ('long time tails'), thus affecting only the small- and large-t regions. Therefore we conclude that the exponential decay law should be valid over a wide intermediate-time region as is well known for unstable quantum systems [19].

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